

MEASURE THEORY IN THE GEOMETRY OF $GL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ DANIELE MUNDICI[†]

ABSTRACT. The n -dimensional affine group over the integers is the group \mathcal{G}_n of all affinities on \mathbb{R}^n which leave the lattice \mathbb{Z}^n invariant. \mathcal{G}_n yields a geometry in the classical sense of the Erlangen Program. In this paper we construct a \mathcal{G}_n -invariant measure on rational polyhedra in \mathbb{R}^n , i.e., finite unions of simplexes with rational vertices in \mathbb{R}^n , and prove its uniqueness. Our main tool is given by the Morelli-Włodarczyk factorization of birational toric maps in blow-ups and blow-downs (solution of the weak Oda conjecture).

1. INTRODUCTION

Following Stallings [10] and Tsujii [11], by a *polyhedron* P in \mathbb{R}^n ($n = 1, 2, \dots$) we mean the pointwise union of a finite set of (always closed) simplexes T_i in \mathbb{R}^n . If the vertices of each T_i are in \mathbb{Q}^n , P is said to be a *rational polyhedron*. P need not be convex or connected: as a matter of fact, every compact subset of \mathbb{R}^n coincides with the intersection of all rational polyhedra in \mathbb{R}^n containing it. We will write $\mathcal{P}^{(n)}$ for the set of all rational polyhedra in \mathbb{R}^n .

We let $\mathcal{G}_n = GL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ denote the group of transformations of the form

$$x \mapsto Ax + t \quad \text{for } x \in \mathbb{R}^n,$$

where $t \in \mathbb{Z}^n$ and A is an $n \times n$ matrix with integer elements and determinant ± 1 . Any such transformation preserves the lattice \mathbb{Z}^n of integer points in \mathbb{R}^n .

As proved in Theorems 4.2 and 8.2, up to a positive multiplicative constant, every rational polyhedron $P \subseteq \mathbb{R}^n$ carries a unique \mathcal{G}_n -invariant d -dimensional rational measure $\lambda_d(P)$, $d = 0, 1, \dots$.

To construct λ_d , writing $(P, 1)$ as an abbreviation of $\{(x, 1) \in \mathbb{R}^{n+1} \mid x \in P\}$, we let Φ be a regular fan over the set given by $\{\theta y \in \mathbb{R}^{n+1} \mid 0 \leq \theta \in \mathbb{R}, y \in (P, 1)\}$. Next we let Δ_Φ be the triangulation of $(P, 1)$ obtained by intersecting every cone of Φ with the hyperplane $x_{n+1} = 1$. Finally, we define

$$\lambda_d(P) = \sum \left\{ \frac{1}{d! \prod_{v \in \text{ext}(T)} \text{den}(v)} \mid T \text{ a maximal } d\text{-simplex of } \Delta_\Phi \right\},$$

where $\text{den}(v)$ denotes the least common denominator of the coordinates of the rational point $v \in \mathbb{R}^{n+1}$, and $\text{ext}(T)$ is the set of vertices of T . As a consequence of the celebrated solution of the weak Oda conjecture by Morelli and Włodarczyk [6, 12], this quantity turns out not to depend on the chosen triangulation Δ_Φ .

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As proved in Corollary 6.2, Lebesgue measure on \mathbb{R}^n is obtainable from λ_n via Carathéodory's construction. However, in contrast to Lebesgue measure, for each $0 \leq d < n$ and d -simplex $T \in \mathcal{P}^{(n)}$, the rational measure $\lambda_d(T)$ does not vanish, and is proportional to the Hausdorff d -dimensional measure of T , with the constant of proportionality only depending on the affine hull of T (see 7.2).

All necessary preliminaries on rational polyhedra (resp., cones) and their regular triangulations (resp., regular, or nonsingular, fans) are collected in the next section.

2. FAREY BLOW-UPS AND BLOW-DOWNS

For any rational point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n = 1, 2, \dots$, we let $\text{den}(x)$ denote the least common denominator of the coordinates of x . The integer vector

$$\tilde{x} = \text{den}(x)(x_1, \dots, x_n, 1) \in \mathbb{Z}^{n+1}$$

is called the *homogeneous correspondent* of x .

For $m = 0, 1, \dots$, an m -simplex $T = \text{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$ is said to be *rational* if all its vertices are rational. We use the notation

$$T^\uparrow = \mathbb{R}_{\geq 0} \tilde{v}_0 + \dots + \mathbb{R}_{\geq 0} \tilde{v}_m \subseteq \mathbb{R}^{n+1} \quad (1)$$

for the positive span in \mathbb{R}^{n+1} of the homogeneous correspondents of the vertices of T . We say that T^\uparrow is the (*rational simplicial*) *cone* of T . The *generators* $\tilde{v}_0, \dots, \tilde{v}_m$ of T^\uparrow are *primitive*, in the sense that each \tilde{v}_i is minimal as a nonzero integer vector along its ray $\mathbb{R}_{\geq 0} \tilde{v}_i$. T^\uparrow uniquely determines the set of its primitive generators, just as T uniquely determines the set $\text{ext}(T)$ of its vertices.

Following [3] we say that T^\uparrow is *regular* if its primitive generators are part of a basis of the free abelian group \mathbb{Z}^{n+1} . By definition, a rational m -simplex $T = \text{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$ is *regular* if so is T^\uparrow .

By a *complex* in \mathbb{R}^n we mean a finite set Λ of compact convex polyhedra P_i in \mathbb{R}^n , closed under taking faces, and having the further property that any two elements of Λ intersect in a common face. Unless otherwise specified, Λ will be *simplicial*, in the sense that all P_i are simplexes.

A simplicial complex Λ is *rational* if all simplexes of Λ are rational: in this case, the set $\Lambda^\uparrow = \{T^\uparrow \mid T \in \Lambda\}$ is a *simplicial fan* [3, 8]. We further say that Λ is *regular* if the fan Λ^\uparrow is regular (= nonsingular in [8]), meaning that every cone $T^\uparrow \in \Lambda^\uparrow$ is regular.

For every complex Λ , its *support* $|\Lambda| \subseteq \mathbb{R}^n$ is the pointset union of all polyhedra of Λ . Instead of saying that Λ is a simplicial complex with support P , we briefly say that Λ is a *triangulation* of P .

Given two simplicial complexes Λ' and Λ with the same support, we say that Λ' is a *subdivision* of Λ if every simplex of Λ' is contained in a simplex of Λ . For any $c \in |\Lambda|$, the *blow-up* $\Lambda_{(c)}$ of Λ at c is the subdivision of Λ given by replacing every simplex $C \in \Lambda$ that contains c by the set of all simplexes of the form $\text{conv}(F \cup \{c\})$, where F is any face of C that does not contain c (see [12, p. 376], [3, III, 2.1]).

The inverse of a blow-up is called a *blow-down*.

For any regular m -simplex $T = \text{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$, the *Farey mediant* of T is the rational point v of T whose homogeneous correspondent \tilde{v} coincides with $\tilde{v}_0 + \dots + \tilde{v}_m$. If T belongs to a regular complex Δ and c is the Farey mediant of T , then the *Farey blow-up* $\Delta_{(c)}$ is regular.

By a *rational (affine) hyperplane* $H \subseteq \mathbb{R}^n$ we mean a subset of \mathbb{R}^n of the form $\{x \in \mathbb{R}^n \mid a \circ x + t = 0\}$, where \circ denotes scalar product, a and t are vectors in \mathbb{Q}^n (equivalently, in \mathbb{Z}^n) and $a \neq 0$. When $t = 0$, H is called *homogeneous*.

By a *rational affine subspace* of \mathbb{R}^n we mean the intersection $A_{\mathcal{F}}$ of a finite set \mathcal{F} of rational hyperplanes in \mathbb{R}^n . In particular, $A_{\emptyset} = \mathbb{R}^n$.

The *affine hull* $\text{aff}(T)$ of a simplex T in \mathbb{R}^n is the set of all affine combinations of points of T .

3. REGULAR TRIANGULATIONS AND \mathbb{Z} -HOMEOMORPHISM

Lemma 3.1. *Every rational polyhedron $P \subseteq \mathbb{R}^n$ is the support of a regular complex.*

Proof. By [10, p.36], P is the support of some simplicial complex Λ . Since P is rational, Λ can be assumed rational. The set $\Lambda^\uparrow = \{T^\uparrow \mid T \in \Lambda\}$ is a simplicial fan in \mathbb{R}^{n+1} . The desingularization procedure of [3, VI, 8.5] yields a regular subdivision Λ^* of Λ^\uparrow . Intersecting each cone of Λ^* with the hyperplane $x_{n+1} = 1$ we obtain a simplicial complex Δ whose support is the set $(P, 1) = \{(x, 1) \in \mathbb{R}^{n+1} \mid x \in P\}$. For each simplex $U \in \Delta$ let U' be the projection of U onto the hyperplane $x_{n+1} = 0$, identified with \mathbb{R}^n . Then the regularity of Λ^* ensures that the set $\{U' \mid U \in \Delta\}$ is a regular complex with support P . \square

The following notion is of independent interest [1, 7], and will find repeated use in this paper.

Definition 3.2. Two rational polyhedra $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ are \mathbb{Z} -*homeomorphic*, $P \cong_{\mathbb{Z}} Q$, if there is a piecewise linear homeomorphism $\eta = (\eta_1, \dots, \eta_m)$ of P onto Q (each η_i with a finite number of pieces $l_{i1}, \dots, l_{ik(i)}$) such that each linear piece of η and η^{-1} is a linear (affine) map with integer coefficients.

In particular, if $m = n$ and there is $\gamma \in \mathcal{G}_n$ with $Q = \gamma(P)$ then $P \cong_{\mathbb{Z}} Q$. The converse does not hold: the two 0-simplexes $\{1/5\}$ and $\{2/5\}$ in \mathbb{R} are \mathbb{Z} -homeomorphic but there is no $\gamma \in \mathcal{G}_1$ such that $\gamma(1/5) = 2/5$.

Lemma 3.3. *Suppose $P \subseteq \mathbb{R}^n$ and $P' \subseteq \mathbb{R}^{n'}$ are rational polyhedra and η is a \mathbb{Z} -homeomorphism of P onto P' .*

- (i) *A point $z \in P$ is rational iff so is the point $\eta(z) \in P'$. Further, $\text{den}(y) = \text{den}(\eta(y))$ for every rational point $y \in P$.*
- (ii) *There is a regular complex Λ with support P such that η is linear (always in the affine sense) over every simplex of Λ .*
- (iii) *For any regular complex Λ with support P such that η is linear over every simplex of Λ , the set $\Lambda' = \{\eta(S) \mid S \in \Lambda\}$ is a regular complex with support P' .*

Proof. (i) Is an immediate consequence of Definition 3.2.

(ii) Lemma 3.1 yields a regular complex \mathcal{C}_0 with support P . Let $\eta_1, \dots, \eta_{n'}$ be the components of η . Fix $i = 1, \dots, n'$ and let l_{i1}, \dots, l_{ik} be the linear pieces of η_i . Letting σ range over all permutations of the set $\{1, \dots, k\}$, the family of sets $P_\sigma = \{x \in P \mid l_{i\sigma(1)} \leq \dots \leq l_{i\sigma(k)}\}$ determines a rational complex \mathcal{C}_i with support P , such that the maps l_{ij} are *stratified* over each polyhedron R of \mathcal{C}_i , in the sense that for all $j' \neq j''$ we either have $l_{ij'} \leq l_{ij''}$ or $l_{ij'} \geq l_{ij''}$ on R . Since every complex can be subdivided into a simplicial complex without adding new vertices, [3, III, 2.6], we can assume without loss of generality that all polyhedra in \mathcal{C}_i are simplexes

and that \mathcal{C}_i is a subdivision of \mathcal{C}_0 . Thus η_i is linear over every simplex of \mathcal{C}_i . One now routinely constructs a common subdivision \mathcal{C} of the rational complexes $\mathcal{C}_1, \dots, \mathcal{C}_{n'}$, such that every simplex of \mathcal{C} is rational. It follows that η is linear over each simplex of \mathcal{C} . The set $\mathcal{C}^\uparrow = \{T^\uparrow \mid T \in \mathcal{C}\}$ is a simplicial fan. The desingularization procedure [3, VI, 8.5] yields a regular fan Φ such that every cone of \mathcal{C}^\uparrow is a union of cones of Φ . Intersecting the cones in Φ with the hyperplane $x_{n+1} = 1$ we have a complex Ξ whose support is the set

$$(P, 1) = \{(x, 1) \in \mathbb{R}^{n+1} \mid x \in P\}.$$

Dropping the last coordinate from the vertices of the simplexes of Ξ we obtain a regular complex Λ with support P such that η is linear over every simplex of Λ .

(iii) Λ' is a rational simplicial complex with support P' . Fix a rational j -simplex $S = \text{conv}(v_0, \dots, v_j) \subseteq P$, not necessarily belonging to Λ , such that η is linear over S . Let $S' = \eta(S)$. The (affine) linear map $\eta: x \in S \mapsto y \in S'$ determines the homogeneous linear map $(x, 1) \in (S, 1) \mapsto (y, 1) \in (S', 1)$. Let M_S be the $(n' + 1) \times (n + 1)$ integer matrix whose bottom row has the form $(0, 0, \dots, 0, 0, 1)$, (with n zeros), and whose i th row ($i = 1, \dots, n'$) is given by the coefficients of the (affine) linear polynomial $\eta_i \upharpoonright S$. Let $\tilde{v}_0, \dots, \tilde{v}_j \in \mathbb{Z}^{n+1}$ be the homogeneous correspondents of the vertices v_0, \dots, v_j of S , and $S^\uparrow = \mathbb{R}_{\geq 0} \tilde{v}_0 + \dots + \mathbb{R}_{\geq 0} \tilde{v}_j \subseteq \mathbb{R}^{n+1}$ be the positive span of $\tilde{v}_0, \dots, \tilde{v}_j$. Let similarly S'^\uparrow be the positive span in $\mathbb{R}^{n'+1}$ of the integer vectors $M_S \tilde{v}_0, \dots, M_S \tilde{v}_j$. By construction, M_S sends integer points of S^\uparrow one-one into integer points of S'^\uparrow . Interchanging the roles of S and S' we see that M_S sends integer points of S'^\uparrow one-one onto integer points of S^\uparrow . Blichfeldt's theorem [2, III.2], yields the following characterization:

$$\begin{aligned} & S \text{ is regular} \\ \Leftrightarrow & \text{the half-open parallelepiped } Q_S = \{\mu_0 \tilde{v}_0 + \dots + \mu_j \tilde{v}_j \mid 0 \leq \mu_0, \dots, \mu_j < 1\} \\ & \text{contains no nonzero integer points} \\ \Leftrightarrow & \text{the half-open parallelepiped } Q_{S'} \text{ contains no nonzero integer points} \\ \Leftrightarrow & S' \text{ is regular.} \end{aligned}$$

In particular, if S is a simplex of Λ then the assumed regularity of Λ entails the regularity of S , whence S' is regular. We conclude that Λ' is a regular complex with support P' . \square

4. LENGTH, AREA, VOLUME IN \mathcal{G}_n -GEOMETRY

For $n > 0$ a fixed integer, let $Q \subseteq \mathbb{R}^n$ be a (not necessarily rational) polyhedron. For any triangulation \mathcal{T} of Q and $i = 0, 1, \dots$ we let $\mathcal{T}^{\max}(i)$ denote the set of maximal i -simplexes of \mathcal{T} . The i -dimensional part $Q^{(i)}$ of Q is now defined by

$$Q^{(i)} = \bigcup \{T \in \mathcal{T}^{\max}(i)\}. \quad (2)$$

Since any two triangulations of Q have a joint subdivision, the definition of $Q^{(i)}$ does not depend on the chosen triangulation \mathcal{T} of Q . If $Q^{(i)}$ is nonempty, then it is an i -dimensional polyhedron whose j -dimensional part $Q^{(j)}$ is empty for each $j \neq i$. Trivially, $Q^{(k)} = \emptyset$ for each integer $k > \dim(Q)$.

For every regular m -simplex $S = \text{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$ we use the notation

$$\text{den}(S) = \prod_{j=0}^m \text{den}(v_j), \quad (3)$$

and say that $\text{den}(S)$ is the *denominator* of S .

For any $P \in \mathcal{P}^{(n)}$, regular triangulation Δ of P , and $i = 0, 1, \dots$, the rational number $\lambda(n, i, P, \Delta)$ is defined by

$$\lambda(n, i, P, \Delta) = \sum_{T \in \Delta^{\max(i)}} \frac{1}{i! \text{den}(T)},$$

with the proviso that $\lambda(n, i, P, \Delta) = 0$ if $\Delta^{\max(i)} = \emptyset$. In particular, this is the case of all $i > \dim(P)$.

Proposition 4.1. *For every $n = 1, 2, \dots$, $i = 0, 1, \dots$, polyhedron $P \in \mathcal{P}^{(n)}$ and regular triangulations Δ and Δ' of P , $\lambda(n, i, P, \Delta) = \lambda(n, i, P, \Delta')$.*

Proof. We first suppose that Δ' is obtained from Δ by a blow-up at the Farey median c of some j -simplex $S = \text{conv}(v_0, \dots, v_j) \in \Delta$, $j = 1, \dots, n$. In symbols, $\Delta' = \Delta_{(c)}$. S is the smallest simplex of Δ containing c as an element. Thus $c \in R \in \Delta \Rightarrow \dim(R) \geq j$. Let $d = 0, 1, \dots, n$. If for no simplex $T \in \Delta^{\max(d)}$ it is the case that $c \in T$, then $\Delta^{\max(d)} = \Delta'^{\max(d)}$. Otherwise, let $T = \text{conv}(v_0, \dots, v_j, \dots, v_d)$ be a simplex of $\Delta^{\max(d)}$ such that $c \in T$. We now define the d -simplexes S_0, \dots, S_j as follows: $S_0 = \text{conv}(c, v_1, \dots, v_d)$, $S_j = \text{conv}(v_0, v_1, \dots, v_{j-1}, c, \dots, v_d)$, and $S_t = \text{conv}(v_0, \dots, v_{t-1}, c, v_{t+1}, \dots, v_j, \dots, v_d)$ for each $t = 1, \dots, j-1$. By definition of Farey median, $\text{den}(c) = \text{den}(v_0) + \dots + \text{den}(v_j)$. By definition of Farey blow-up, the subcomplex of Δ given by T and its faces is replaced in Δ' by the simplicial complex given by the d -simplexes S_0, \dots, S_j and their faces. Since T is regular, then so is S_u for each $u = 0, \dots, j$, whence $\text{den}(S_u) = \text{den}(T) \cdot \text{den}(c) / \text{den}(v_u)$. As a consequence, $1/\text{den}(T) = \sum_{u=0}^j 1/\text{den}(S_u)$. Since

$$\sum_{T \in \Delta^{\max(d)}} \frac{1}{d! \text{den}(T)} = \sum_{U \in \Delta'^{\max(d)}} \frac{1}{d! \text{den}(U)},$$

then $\lambda(n, d, P, \Delta) = \lambda(n, d, P, \Delta')$. Thus in case $\Delta' = \Delta_{(c)}$ we get $\lambda(n, i, P, \Delta) = \lambda(n, i, P, \Delta')$, for all $i = 0, 1, \dots$.

In the general case when Δ' is an arbitrary regular triangulation of P , the solution of the weak Oda conjecture [6, 12] yields a sequence of regular triangulations $\nabla_0 = \Delta$, $\nabla_1, \dots, \nabla_{s-1}$, $\nabla_s = \Delta'$, where each ∇_{k+1} is obtained from ∇_k by a Farey blow-up, or vice versa, ∇_k is obtained from ∇_{k+1} by a Farey blow-up. Then the desired conclusion follows by induction on s . \square

In the light of the foregoing proposition, for each $n = 1, 2, \dots$, polyhedron $P \in \mathcal{P}^{(n)}$ and $d = 0, 1, \dots$, we can unambiguously write

$$\lambda_d(P) = \lambda(n, d, P, \Delta), \quad (4)$$

where Δ is an arbitrary regular triangulation of P . We say that λ_d is the d -dimensional *rational measure* of P . Trivially, $\lambda_d(P) = 0$ for each integer $d > \dim(P)$. If clarity requires it, different symbols λ_d and λ'_d may be used, e.g., for the d -dimensional rational measures defined on $\mathcal{P}^{(n)}$ and $\mathcal{P}^{(n+1)}$. However, no such

notational distinction will be necessary when the ambient space \mathbb{R}^n is clear from the context.

Theorem 4.2. *For each $n = 1, 2, \dots$ and $d = 0, 1, \dots$, the map $\lambda_d: \mathcal{P}^{(n)} \rightarrow \mathbb{R}_{\geq 0}$ has the following properties, for all $P, Q \in \mathcal{P}^{(n)}$:*

- (i) **Invariance:** *If $P = \gamma(Q)$ for some $\gamma \in \mathcal{G}_n$ then $\lambda_d(P) = \lambda_d(Q)$.*
- (ii) **Valuation:** *$\lambda_d(\emptyset) = 0$, $\lambda_d(P) = \lambda_d(P^{(d)})$, and the restriction of λ_d to the set of all rational polyhedra P, Q in \mathbb{R}^n having dimension at most d is a valuation: in other words,*

$$\lambda_d(P) + \lambda_d(Q) = \lambda_d(P \cup Q) + \lambda_d(P \cap Q). \quad (5)$$

- (iii) **Conservativity:** *For any $P \in \mathcal{P}^{(n)}$ let $(P, 0) = \{(x, 0) \in \mathbb{R}^{n+1} \mid x \in P\}$. Then $\lambda_d(P) = \lambda_d(P, 0)$.*
- (iv) **Pyramid:** *For $k = 1, \dots, n$, if $\text{conv}(v_0, \dots, v_k)$ is a regular k -simplex in \mathbb{R}^n with $v_0 \in \mathbb{Z}^n$ then*

$$\lambda_k(\text{conv}(v_0, \dots, v_k)) = \lambda_{k-1}(\text{conv}(v_1, \dots, v_k))/k. \quad (6)$$

- (v) **Normalization:** *Let $j = 1, \dots, n$. Suppose the set $B = \{w_1, \dots, w_j\} \subseteq \mathbb{Z}^n$ is part of a basis of the free abelian group \mathbb{Z}^n . Let the closed parallelepiped $P_B \subseteq \mathbb{R}^n$ be defined by*

$$P_B = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^j \gamma_i w_i, \ 0 \leq \gamma_i \leq 1 \right\}. \quad (7)$$

Then $\lambda_j(P_B) = 1$.

- (vi) **Proportionality:** *Let A be an m -dimensional rational affine subspace of \mathbb{R}^n for some $m = 0, \dots, n$. Then there is a constant $\kappa_A > 0$, only depending on A , such that $\lambda_m(Q) = \kappa_A \cdot \mathcal{H}^m(Q)$ for every rational m -simplex $Q \subseteq A$. Here as usual, \mathcal{H}^m denotes m -dimensional Hausdorff measure.*

In Theorem 8.2 below we will prove that conditions (i)-(vi) uniquely characterize the maps $\lambda_d: \mathcal{P}^{(n)} \rightarrow \mathbb{R}_{\geq 0}$. We first prove that conditions (i)-(vi) hold.

5. PROOF OF THEOREM 4.2(i)-(v)

5.1. Invariance. We will actually prove the stronger result that λ_d is invariant under \mathbb{Z} -homeomorphisms: in other words, whenever $P' \subseteq \mathbb{R}^{n'}$ is a rational polyhedron and $P \cong_{\mathbb{Z}} P'$ then $\lambda_d(P) = \lambda_d(P')$ for all $d = 0, 1, \dots$. Let ι be a \mathbb{Z} -homeomorphism of P onto P' . Let Δ be a regular complex with support P such that ι is (affine) linear over every simplex of Δ . The existence of Δ is ensured by Lemma 3.3(ii). Let $\Delta' = \{\iota(T) \mid T \in \Delta\}$. By Lemma 3.3(i)-(iii), Δ' is a regular complex with support P' , and $\text{den}(\iota(z)) = \text{den}(z)$ for every rational point $z \in P$. It follows that $\lambda(n, d, P, \Delta) = \lambda(n', d, P', \Delta')$. The desired conclusion now follows from (4) as a consequence of Proposition 4.1.

5.2. Valuation. The identities $\lambda_d(\emptyset) = 0$, and $\lambda_d(P) = \lambda_d(P^{(d)})$ immediately follow by definition of rational measure. To prove (5), we first observe that both $P \cup Q$ and $P \cap Q$ are rational polyhedra in \mathbb{R}^n whose dimension is at most d . As an application of Lemma 3.1, let the regular complexes $\Delta, \Phi, \Psi, \Omega$ have the following properties:

$$|\Delta| = P \cap Q, \quad |\Phi| = P, \quad |\Psi| = Q, \quad |\Omega| = P \cup Q.$$

Using the extension argument in [3, VI. 9.3] we can assume $\Delta = \Phi \cap \Psi$ and $\Omega = \Phi \cup \Psi$, without loss of generality. For every $X \subseteq \mathbb{R}^n$ we let as usual $\text{cl}(X)$ denote the closure of X in \mathbb{R}^n . By Proposition 4.1 we have:

$$\begin{aligned} \lambda_d(P) + \lambda_d(Q) &= \lambda(n, d, P, \Phi) + \lambda(n, d, Q, \Psi) \\ &= \frac{1}{d!} \left[\sum_{T \in \Phi^{\max}(d)} \text{den}(T)^{-1} + \sum_{T \in \Psi^{\max}(d)} \text{den}(T)^{-1} \right] \\ &= \frac{1}{d!} \left[\sum_{\text{cl}(P \setminus Q) \supseteq T \in \Phi^{\max}(d)} \text{den}(T)^{-1} + \sum_{\text{cl}(Q \setminus P) \supseteq T \in \Psi^{\max}(d)} \text{den}(T)^{-1} \right] \\ &\quad + \frac{1}{d!} \left[\sum_{P \cap Q \supseteq T \in \Phi^{\max}(d)} \text{den}(T)^{-1} + \sum_{P \cap Q \supseteq T \in \Psi^{\max}(d)} \text{den}(T)^{-1} \right] \\ &= \frac{1}{d!} \left[\sum_{\text{cl}(P \setminus Q) \supseteq T \in \Phi^{\max}(d)} \text{den}(T)^{-1} + \sum_{\text{cl}(Q \setminus P) \supseteq T \in \Psi^{\max}(d)} \text{den}(T)^{-1} \right] \\ &\quad + \frac{2}{d!} \sum_{T \in \Delta^{\max}(d)} \text{den}(T)^{-1} \\ &= \frac{1}{d!} \left[\sum_{\text{cl}(P \setminus Q) \supseteq T \in \Omega^{\max}(d)} \text{den}(T)^{-1} + \sum_{\text{cl}(Q \setminus P) \supseteq T \in \Omega^{\max}(d)} \text{den}(T)^{-1} \right] \\ &\quad + \frac{2}{d!} \sum_{T \in \Delta^{\max}(d)} \text{den}(T)^{-1} \\ &= \frac{1}{d!} \left[\sum_{\text{cl}(P \setminus Q) \supseteq T \in \Omega^{\max}(d)} \text{den}(T)^{-1} + \sum_{\text{cl}(Q \setminus P) \supseteq T \in \Omega^{\max}(d)} \text{den}(T)^{-1} \right] \\ &\quad + \frac{1}{d!} \left[\sum_{P \cap Q \supseteq T \in \Omega^{\max}(d)} \text{den}(T)^{-1} + \sum_{T \in \Delta^{\max}(d)} \text{den}(T)^{-1} \right] \\ &= \lambda(n, d, P \cup Q, \Omega) + \lambda(n, d, P \cap Q, \Delta) = \lambda_d(P \cup Q) + \lambda_d(P \cap Q). \end{aligned}$$

5.3. Conservativity and Pyramid. Properties (iii)-(iv) are immediate consequences of the definition of λ_d .

5.4. Normalization. To prove Property (v), let Π be the set of permutations of the set $\{1, 2, \dots, j\}$. For every permutation $\pi \in \Pi$ we let T_π be the convex hull of the set of points

$$0, w_{\pi(1)}, w_{\pi(1)} + w_{\pi(2)}, w_{\pi(1)} + w_{\pi(2)} + w_{\pi(3)}, \dots, w_{\pi(1)} + w_{\pi(2)} + \dots + w_{\pi(j)}.$$

Arguing as in [9, 3.4], it follows that the j -simplexes T_π are the maximal elements of a triangulation Σ of P_B , called the *standard triangulation* Σ . Each simplex T_π is regular and has unit denominator. There are $j!$ such simplexes. By definition, the rational j -dimensional measure of T_π is equal to $1/j!$. A final application of 4.1 yields $\lambda_j(P_B) = 1$.

6. FROM λ_n TO LEBESGUE MEASURE ON \mathbb{R}^n VIA CARATHÉODORY'S METHOD

In what follows, \mathcal{L}^n will denote Lebesgue measure on \mathbb{R}^n .

Proposition 6.1. *For any $n = 1, 2, \dots$ and polyhedron $Q \in \mathcal{P}^{(n)}$, $\lambda_n(Q) = \mathcal{L}^n(Q)$.*

Proof. If $\dim(Q) < n$ then $\mathcal{L}^n(Q) = \lambda_n(Q) = 0$. If $\dim(Q) = n$, since $\lambda_n(Q) = \lambda_n(Q^{(n)})$ and $\mathcal{L}^n(Q) = \mathcal{L}^n(Q^{(n)})$, without loss of generality we may assume $Q = Q^{(n)}$. Let ∇ be a regular triangulation of Q as given by Lemma 3.1. Since, as we have seen, λ_n is a valuation on $\mathcal{P}^{(n)}$ and $\mathcal{L}^n(Q) = \sum_{S \in \nabla^{\max(n)}} \mathcal{L}^n(S)$, it is enough to prove

$$\lambda_n(S) = \mathcal{L}^n(S) \text{ for every } n\text{-simplex } S = \text{conv}(w_0, \dots, w_n) \in \nabla. \quad (8)$$

To this purpose, let $T \subseteq \mathbb{R}^{n+1}$ be the $(n+1)$ -simplex with vertices $0, (w_0, 1), \dots, (w_n, 1)$. Then

$$\mathcal{L}^{n+1}(T) = \mathcal{L}^n(S)/(n+1).$$

This is the classical formula for the volume of the $(n+1)$ -dimensional pyramid with base S and height 1. Next we observe that T is contained in the closed $(n+1)$ -dimensional parallelepiped $E = \{\alpha_0(w_0, 1) + \dots + \alpha_n(w_n, 1) \in \mathbb{R}^{n+1} \mid \alpha_0, \dots, \alpha_n \in [0, 1]\}$. Further, $E \subseteq U = \{\alpha_0 \tilde{w}_0 + \dots + \alpha_n \tilde{w}_n \in \mathbb{R}^{n+1} \mid \alpha_0, \dots, \alpha_n \in [0, 1]\}$. Since S is regular, a classical argument in the geometry of numbers, ([2] or [3, Proof of VI, 8.5]) yields $\mathcal{L}^{n+1}(U) = 1$. For all $i = 0, \dots, n$ let $d_i = \text{den}(w_i)$. Since $\tilde{w}_0 = d_0(w_0, 1), \dots, \tilde{w}_n = d_n(w_n, 1)$, then

$$\mathcal{L}^{n+1}(E) = (d_0 \dots d_n)^{-1}.$$

The construction of [9, 3.4] now yields a triangulation of E consisting of $(n+1)$ -simplexes $T_1, \dots, T_{(n+1)!}$ and their faces, in such a way that

$$\mathcal{L}^{n+1}(T_i) = \frac{\mathcal{L}^{n+1}(E)}{(n+1)!} \text{ for each } i = 1, \dots, (n+1)!$$

Each T_i is a regular simplex. One easily gets a linear (affine) isometry of T_i onto T . Therefore,

$$\mathcal{L}^{n+1}(T) = \frac{\mathcal{L}^{n+1}(E)}{(n+1)!}.$$

Summing up, $\mathcal{L}^n(S) = \mathcal{L}^{n+1}(E)/n! = (n! d_0 \dots d_n)^{-1} = \lambda_n(S)$, and (8) is proved. \square

Corollary 6.2. *Fix $n = 1, 2, \dots$ and let $\mathcal{K}^{(n)}$ denote the family of compact subsets of \mathbb{R}^n . For any Borel set $E \subseteq \mathbb{R}^n$ let us define*

$$\bar{\lambda}_n(E) = \sup_{E \supseteq K \in \mathcal{K}^{(n)}} \inf_{K \subseteq P \in \mathcal{P}^{(n)}} \lambda_n(P).$$

Then $\bar{\lambda}_n(E) = \mathcal{L}^n(E)$.

Proof. We first *claim* that every $K \in \mathcal{K}^{(n)}$ coincides with the intersection of all rational polyhedra of $\mathcal{P}^{(n)}$ containing it.

As a matter of fact, for any $P, Q \in \mathcal{P}^{(n)}$ both $P \cup Q$ and $P \cap Q$ are members of $\mathcal{P}^{(n)}$. Moreover, there exists a rational triangulation \mathcal{T} of $P \cup Q$ such that the set $\{T \in \mathcal{T} \mid T \subseteq P \cap Q\}$ is a triangulation of $P \cap Q$. Thus the set $\{T \in \mathcal{T} \mid T \subseteq \text{cl}(P \setminus Q)\}$ is a triangulation of the set $\text{cl}(P \setminus Q) \subseteq \mathbb{R}^n$, which shows that $\text{cl}(P \setminus Q)$ is a rational polyhedron. For every $x \in \mathbb{R}^n \setminus K$ there is a rational n -simplex T

containing x in its interior and such that $T \cap K = \emptyset$. Since K is contained in some rational polyhedron, our claim is settled.

Now let $P_0 \supseteq P_1 \supseteq \dots$ be a sequence of rational polyhedra such that $\bigcap_i P_i = K$ and for every $R \in \mathcal{P}^{(n)}$ with $K \subseteq R$ there is $j = 0, 1, \dots$ such that $P_j \subseteq R$. The existence of this sequence follows from our claim together with the observation that there are only countably many rational polyhedra. By Proposition 6.1, $\lambda_n(P_0) = \mathcal{L}^n(P_0) \geq \mathcal{L}^n(P_1) = \lambda_n(P_1) \geq \lambda_n(P_2) \geq \dots$, whence by construction, $\lim_{i \rightarrow \infty} \lambda_n(P_i) = \inf\{\lambda_n(R) \mid R \supseteq K, R \in \mathcal{P}^{(n)}\} = \bar{\lambda}_n(K)$. Combining Proposition 6.1 with the countable monotonicity property of \mathcal{L}^n , we get $\mathcal{L}^n(K) = \lim_{i \rightarrow \infty} \mathcal{L}^n(P_i) = \lim_{i \rightarrow \infty} \lambda_n(P_i) = \bar{\lambda}_n(K)$.

Having thus proved that $\bar{\lambda}_n$ agrees with \mathcal{L}^n on all compact subsets of \mathbb{R}^n , the desired conclusion follows from the regularity properties of Lebesgue measure. \square

Remark 6.3. Following [5, 115C] we now routinely extend $\bar{\lambda}_n$ to an outer measure $\lambda_n^*: \text{powerset}(\mathbb{R}^n) \rightarrow [0, \infty]$ which, by Corollary 6.2 and [5, 115D] coincides with Lebesgue outer measure on \mathbb{R}^n . As proved in [5, 115E], by applying to λ_n^* Carathéodory's construction [5, 113] we finally obtain Lebesgue measure on \mathbb{R}^n .

7. PROOF OF THEOREM 4.2(VI)

7.1. Basic material on Hausdorff measure. In the following proposition we collect a number of well known consequences of the isodiametric inequality (see [4, 2.10.33]), and of the invariance of Hausdorff d -dimensional measure under isometries:

Proposition 7.1. *For each $0 < n \in \mathbb{Z}$ we have:*

- (i) *If $T = \text{conv}(x_0, \dots, x_n)$ is an n -simplex in \mathbb{R}^n , letting M be the $n \times n$ matrix whose i th row is given by the vector $x_i - x_0$, ($i = 1, \dots, n$), it follows that $\mathcal{H}^n(T) = |\det(M)|/n! = \mathcal{L}^n(T)$.*
- (ii) *If S is an m -simplex in \mathbb{R}^n with $0 < m < n$, and we map S onto a copy S' by means of an isometry ι sending the affine hull of S onto the linear subspace \mathbb{R}^m of \mathbb{R}^n spanned by the first m standard basis vectors of \mathbb{R}^n , then $\mathcal{H}^m(S) = \mathcal{L}^m(S')$. If $\dim(S) = 0$ then $\mathcal{H}^0(S) = 1 = \text{number of elements of the singleton } S$.*
- (iii) *If Q is a nonempty polyhedron in \mathbb{R}^n and $Q = Q^{(d)}$, $d = 0, 1, \dots$, then letting \mathcal{T} be an arbitrary triangulation of Q , with its d -simplexes T_1, \dots, T_k , we have $\mathcal{H}^d(Q) = \sum_{j=1}^k \mathcal{H}^d(T_j)$. If $Q = \emptyset$ then $\mathcal{H}^k(Q) = 0$ for all $k = 0, 1, \dots$.*
- (iv) *Given integers $0 \leq m < n$, suppose $T = \text{conv}(v_0, \dots, v_m)$ and $T' = \text{conv}(v'_0, \dots, v'_m)$ are m -simplexes in \mathbb{R}^n with $\text{aff}(T) = \text{aff}(T')$. For v an arbitrary point lying in $\mathbb{R}^n \setminus \text{aff}(T)$, let $U = \text{conv}(T, v)$ and $U' = \text{conv}(T', v)$. Then $\mathcal{H}^{m+1}(U')/\mathcal{H}^{m+1}(U) = \mathcal{H}^m(T')/\mathcal{H}^m(T)$.*
- (v) *More generally, suppose the points $v_{m+1}, \dots, v_n \in \mathbb{R}^n$ have the property that $W = \text{conv}(v_0, \dots, v_m, v_{m+1}, \dots, v_n)$ is an n -simplex. Then also $W' = \text{conv}(v'_0, \dots, v'_m, v_{m+1}, \dots, v_n)$ is an n -simplex, and we have the identity $\mathcal{H}^n(W')/\mathcal{H}^n(W) = \mathcal{H}^m(T')/\mathcal{H}^m(T)$.*

7.2. End of the proof of Theorem 4.2(vi). There remains to be proved that λ_d has the Proportionality property (vi). By Lemma 3.1, Q has a regular triangulation. Since λ_m is a valuation, recalling Proposition 7.1(iii) it suffices to argue in case Q is a regular m -simplex. If $m = n$ the result follows from Proposition 6.1 since, by Proposition 7.1(i), $\mathcal{H}^n(Q) = \mathcal{L}^n(Q)$. In this case $\kappa_A = 1$. Next suppose $0 \leq m < n$. It suffices to prove that for any two regular m -simplexes $T = \text{conv}(v_0, \dots, v_m)$ and $T' = \text{conv}(v'_0, \dots, v'_m)$ lying in A ,

$$\lambda_m(T)/\lambda_m(T') = \mathcal{H}^m(T)/\mathcal{H}^m(T').$$

To this purpose let $U = \text{conv}(v_0, \dots, v_m, v_{m+1}, \dots, v_n)$ be a regular n -simplex in \mathbb{R}^n having T as a face.

Claim. The simplex $U' = \text{conv}(v'_0, \dots, v'_m, v_{m+1}, \dots, v_n)$ is regular.

As a matter of fact, the regularity of T means that the set $\{\tilde{v}_0, \dots, \tilde{v}_m\}$ is a basis of the free abelian group $G = \mathbb{Z}^{n+1} \cap (\mathbb{R}\tilde{v}_0 + \dots + \mathbb{R}\tilde{v}_m)$ of integer points in the $(m+1)$ -dimensional linear space spanned by $\tilde{v}_0, \dots, \tilde{v}_m$ in \mathbb{R}^{n+1} . Since $\text{aff}(T') = A = \text{aff}(T)$ and T' is regular, also $\tilde{v}'_0, \dots, \tilde{v}'_m$ constitute a basis of G . Upon writing each \tilde{v}_i and \tilde{v}'_j as a column vector, let M be the $(n+1) \times (m+1)$ matrix whose i th row coincides with \tilde{v}_i . Let similarly M' be the $(n+1) \times (m+1)$ matrix whose j th row equals \tilde{v}'_j . Let the $(m+1) \times (m+1)$ integer matrix Z be defined by $MZ = M'$. The $(m+1) \times (m+1)$ integer matrix V defined by $M'V = M$ coincides with Z^{-1} , whence $|\det(Z)| = |\det(Z^{-1})| = 1$. Let the matrix N be defined by

$$N = \left(\begin{array}{c|c} Z & 0 \\ \hline 0 & I_{n-m} \end{array} \right)$$

where I_{n-m} denotes the $(n-m) \times (n-m)$ identity matrix. N is a unimodular integer $(n+1) \times (n+1)$ matrix. Let W (resp., W') be the $(n+1) \times (n+1)$ integer matrix whose first $m+1$ columns are those of M (resp., those of M'), and whose last $n-m$ columns are given by the column vectors $\tilde{v}_{m+1}, \dots, \tilde{v}_n$. From $WN = W'$, it follows that the vectors $\tilde{v}'_0, \dots, \tilde{v}'_m, \tilde{v}_{m+1}, \dots, \tilde{v}_n$ constitute a basis of the free abelian group \mathbb{Z}^{n+1} . Therefore, $\text{conv}(v'_0, \dots, v'_m, v_{m+1}, \dots, v_n)$ is a regular n -simplex in \mathbb{R}^n , and our claim is settled.

Let now $d_i = \text{den}(v_i)$, ($i = 0, \dots, n$) and $d'_j = \text{den}(v'_j)$, ($j = 0, \dots, m$). Since both simplexes U and U' are regular we can write the identities

$$\frac{\lambda_m(T)}{\lambda_m(T')} = \frac{(m! d_0 \cdots d_m)^{-1}}{(m! d'_0 \cdots d'_m)^{-1}} = \frac{(n! d_0 \cdots d_m d_{m+1} \cdots d_n)^{-1}}{(n! d'_0 \cdots d'_m d_{m+1} \cdots d_n)^{-1}} = \frac{\lambda_n(U)}{\lambda_n(U')}.$$

By Propositions 6.1 and 7.1(ii)-(v) we obtain

$$\frac{\lambda_n(U)}{\lambda_n(U')} = \frac{\mathcal{L}^n(U)}{\mathcal{L}^n(U')} = \frac{\mathcal{H}^n(U)}{\mathcal{H}^n(U')} = \frac{\mathcal{H}^m(T)}{\mathcal{H}^m(T')},$$

as required to prove (vi).

The proof of Theorem 4.2 is now complete. \square

8. UNIQUENESS

For every nonempty rational affine subspace F of \mathbb{R}^n let the integer $d_F \geq 1$ be defined by

$$d_F = \min\{q \in \mathbb{Z} \mid q = \text{den}(r) \text{ for some rational point } r \in F\}. \quad (9)$$

Lemma 8.1. *Fix $n = 1, 2, \dots$ and $e = 0, \dots, n$. Let F be a rational e -dimensional affine subspace of \mathbb{R}^n and $d = d_F$.*

- (i) *There are rational points $v_0, \dots, v_e \in F$, all with denominator d , such that $\text{conv}(v_0, \dots, v_e)$ is a regular e -simplex.*
- (ii) *For any rational point $y \in F$ there is an integer $k = 1, 2, \dots$ such that $\text{den}(y) = kd$.*

Proof. (i) For some regular e -simplex $S_0 = \text{conv}(u_0, \dots, u_e)$ we can write $F = \text{aff}(u_0, \dots, u_e)$. The regularity of S_0 means that the set $B_0 = \{\tilde{u}_0, \dots, \tilde{u}_e\}$ can be extended to a basis of the free abelian group \mathbb{Z}^{n+1} , whence B_0 is a basis of the lattice $\mathbb{Z}^{n+1} \cap F^*$, where $F^* = \mathbb{R}\tilde{u}_0 + \dots + \mathbb{R}\tilde{u}_e$ is the linear subspace of \mathbb{R}^{n+1} generated by $\tilde{u}_0, \dots, \tilde{u}_e$.

It is impossible that the *heights* (=last coordinates) of $\tilde{u}_0, \dots, \tilde{u}_e$ are all equal to an integer $h > d$: for otherwise, any primitive vector \tilde{r} in F^* of height d , for r as in (9), could not arise as a linear combination of the \tilde{u}_i with integer coefficients—and B_0 would not be a basis of $\mathbb{Z}^{n+1} \cap F^*$.

If the heights of $\tilde{u}_0, \dots, \tilde{u}_e$ are all equal to d we have nothing to prove. Otherwise, we will construct a finite sequence B_0, B_1, \dots of bases of $\mathbb{Z}^{n+1} \cap F^*$, and finally obtain a basis $\{\tilde{v}_0, \dots, \tilde{v}_e\}$ having the property that the height of each \tilde{v}_i is equal to d .

The first step is as follows: Choose a vector $\tilde{u}_i \in B_0$ of top height, a vector $\tilde{u}_j \in B_0$ of smaller height, and replace \tilde{u}_i by $\tilde{u}_i - \tilde{u}_j$. We get a new basis B_1 of $\mathbb{Z}^{n+1} \cap F^*$, and a new regular e -simplex S_1 in F . Specifically, letting the rational point $w \in F$ be defined by $\tilde{w} = \tilde{u}_i - \tilde{u}_j$, the vertices of S_1 are $u_0, \dots, u_{i-1}, w, u_{i+1}, \dots, u_e$. Observe that the sum of the heights of the elements of B_1 is strictly smaller than the sum of the heights of the elements of B_0 .

Proceeding inductively, and replacing a top vector \tilde{u} of the basis B_t by a vector $\tilde{u} - \tilde{v}$ with $\tilde{v} \in B_t$ of smaller height than \tilde{u} , we obtain a new basis B_{t+1} such that the sum of the heights of the elements of B_{t+1} is strictly smaller than the sum of the heights of the elements of B_t . We also get a new regular e -simplex S_{t+1} lying in F . The process must terminate with a basis $\{\tilde{v}_0, \dots, \tilde{v}_e\}$ of $\mathbb{Z}^{n+1} \cap F^*$ where all \tilde{v}_i have the same height—which by our initial discussion must be equal to d . By definition, $\text{conv}(v_0, \dots, v_e)$ is the desired regular e -simplex in F .

(ii) now trivially follows from (i): for, the regularity of $\text{conv}(v_0, \dots, v_e)$ is to the effect that the primitive vector $\tilde{y} \in \mathbb{Z}^{n+1}$ is a linear combination of the \tilde{v}_i with integer coefficients. \square

Theorem 8.2. *Properties (i)-(vi) in Theorem 4.2 uniquely characterize the rational measures $\lambda_0, \dots, \lambda_n$, for each $n = 1, 2, \dots$, among all maps from $\mathcal{P}^{(n)}$ to $\mathbb{R}_{\geq 0}$.*

Proof. Suppose for each $n = 1, 2, \dots$, the maps $\mu_0, \dots, \mu_n: \mathcal{P}^{(n)} \rightarrow \mathbb{R}_{\geq 0}$, as well as the maps $\mu'_0, \dots, \mu'_{n+1}: \mathcal{P}^{(n+1)} \rightarrow \mathbb{R}_{\geq 0}$ have all properties (i)-(vi). Since by Lemma 3.1 every rational polyhedron has a regular triangulation, and each μ_j and λ_j is a valuation, it suffices to show that $\mu_m(S) = \lambda_m(S)$ for all $m = 0, \dots, n$, and regular

m -simplex S in \mathbb{R}^n . Let $F = \text{aff}(S)$ be the affine hull of S in \mathbb{R}^n . Let $d = d_F$ be the smallest denominator of a rational point of F as in (9) above. Let us identify \mathbb{R}^n with the hyperplane $x_{n+1} = 0$ of \mathbb{R}^{n+1} . Let $T = \text{conv}(v_0, \dots, v_m) \subseteq F$ be a regular m -simplex such that $\text{den}(v_0) = \dots = \text{den}(v_m) = d$. The existence of T is ensured by Lemma 8.1. Let $T' = \{(x, 1) \in \mathbb{R}^{n+1} \mid x \in T\}$. There is $\alpha \in \mathcal{G}_{n+1}$ such that $\alpha(T, 0) = T'$. From the Invariance and Conservativity properties of all μ_i and μ'_j we obtain

$$\mu'_m(T') = \mu'_m(T, 0) = \mu_m(T). \quad (10)$$

The regularity of T means that the set $B = \{\tilde{v}_0, \dots, \tilde{v}_m\}$ is part of a basis of the free abelian group \mathbb{Z}^{n+1} . As in (7) above, let the closed parallelepiped P_B be defined by

$$P_B = \left\{ x \in \mathbb{R}^{n+1} \mid x = \sum_{i=0}^m \gamma_i \tilde{v}_i, \ 0 \leq \gamma_i \leq 1 \right\}.$$

From the Normalization property we get $\mu'_{m+1}(P_B) = 1$. Arguing as in 5.4 above, we obtain a triangulation Δ of P_B consisting of $(m+1)$ -simplexes $T_1, \dots, T_{(m+1)!}$ and their faces. Each T_i is regular and has denominator 1. A direct verification shows that for any two such simplexes T_i and T_j there is $\gamma \in \mathcal{G}_{n+1}$ such that $T_i = \gamma(T_j)$. From the Valuation and Invariance properties of μ'_{m+1} it follows that

$$\mu'_{m+1}(T_j) = \frac{\mu'_{m+1}(P_B)}{(m+1)!} = \frac{1}{(m+1)!} \quad \text{for all } j = 1, \dots, (m+1)!$$

Let $D \subseteq \mathbb{R}^{n+1}$ be the $(m+1)$ -simplex with vertices $0, \tilde{v}_0, \dots, \tilde{v}_m$. It is easily seen that D is regular and $\text{den}(D) = 1$. Thus an easy exercise yields an $\eta \in \mathcal{G}_{n+1}$ such that $\eta(T_1) = D$. One more application of the Invariance property of μ'_{m+1} yields

$$\mu'_{m+1}(D) = \frac{1}{(m+1)!}.$$

Since the $(m+1)$ -simplex D' with vertices $0, (v_0, 1), \dots, (v_m, 1)$ has the same affine hull as D , by the assumed Proportionality property of μ'_{m+1} we have

$$\mu'_{m+1}(D') = \frac{1}{(m+1)! d^{m+1}}.$$

On the other hand, the Pyramid property is to the effect that

$$\mu'_{m+1}(D') = \frac{\mu'_m(T')}{m+1},$$

whence

$$\mu'_m(T') = \frac{1}{m! d^{m+1}}.$$

Recalling (10) we get

$$\mu_m(T) = \frac{1}{m! d^{m+1}} = \lambda_m(T),$$

because T is regular and all the denominators of its vertices are equal to d . Since S and T have the same affine hull, a final application of the Proportionality property of μ_m and λ_m yields

$$\frac{\mu_m(S)}{\mu_m(T)} = \frac{\mathcal{H}^m(S)}{\mathcal{H}^m(T)} = \frac{\lambda_m(S)}{\lambda_m(T)}.$$

In conclusion,

$$\mu_m(S) = \lambda_m(S) \frac{\mu_m(T)}{\lambda_m(T)} = \lambda_m(S).$$

The proof is complete. □

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